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Location of the Zeros of Polynomials Satisfying Three-Term Recurrence Relations. III. Positive Coefficients Case

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The location of the zeros of a family of polynomials satisfying a three-term recurrence relation are studied. Various cases where some of the coefficients are positive or merely real (but the main coefficients are positive) are considered, and new results are obtained.

INTRODUCTION

In the papers [2, 3], we have studied the location of the zeros of a family of polynomials satisfying a three-term recurrence relation with complex coefficients. This work has interesting applications in approximation theory. For instance, in the study of zeros and poles of general Padé approximants or the location of the zeros of general orthogonal polynomials.

Here we shall complement our results in the case of real coefficients; more particularly, in the case of positive coefficients. Many authors have studied this problem in this particular case, e.g., Saff and Varga [5], whose purpose was to study the distribution of the zeros of some special polynomials [1].

In the present paper we shall give some improvements of previous results (cf. Theorem 1.1). We shall also present two new results which allow, in particular, analysis of the distribution of the zeros and poles of a family of Padé approximants to the functions of class S [4] (cf. Theorem 2.1) and the zeros of a family of orthogonal polynomials (cf. Theorem 2.2).

To carry this out, we shall use the properties of homographic transformations.

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1. Notation and Method

Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of polynomials of respective degrees n which satisfy a three-term recurrence relation:

$$P_{n+1}(z) = B_n(z) P_n(z) - A_n(z) P_{n-1}(z) \qquad (n = 0, 1, \ldots), \tag{1.1}$$

where: $P_{-1} \equiv 0, P_0 \equiv p_0 \neq 0$ and the given polynomials B_n $(n = 0, 1, \ldots)$ are of degree 1 while the given polynomials A_n are of degrees $\leqslant 2$ $(n = 1, 2, \ldots)$, with $A_0 \equiv 1$. At present we suppose that A_n and B_n are polynomials with complex coefficients.

If for some fixed integer n and complex z the following conditions are satisfied:

$$P_n(z) \neq 0, \qquad P_{n+1}(z) \neq 0; \tag{1.2}$$

then the relation (1.1) can be written:

$$A_{n+1} \frac{P_n}{P_{n+1}} = \frac{A_{n+1}}{B_n - A_n(P_{n-1}/P_n)}. \tag{1.3}$$

Setting

$$t_n = A_n \frac{P_{n-1}}{P_n} \qquad (n = 0, 1, \ldots), \tag{1.4}$$

we can consider the relation (1.3) as an homographic transformation T_n:

$$t_{n+1} = T_n(t_n), \tag{1.5}$$

where

$$T_n(w) = \frac{A_{n+1}}{B_n - w} \tag{1.6}$$

and

$$A_n(z) \neq 0 \qquad (n = 1, 2, \ldots). \tag{1.7}$$

For the analysis of the transformation (1.6), we shall use only some global information on the poles B_n. To do that, we shall use the well-known property that the transformation T_n maps any half-plane: $\{w \in \mathbb{C} \mid \operatorname{Re} w \lesseqgtr \gamma\}$, (respectively $\{w \in \mathbb{C} \mid \operatorname{Im} w \lesseqgtr \gamma\}$) which does not contain the pole B_n into the closed disc of radius ρ_n centered in ω_n:

$$\omega_n = \frac{A_{n+1}}{2(\operatorname{Re} B_n - \gamma)} \left(\text{resp. } \omega_n = \frac{-iA_{n+1}}{2(\operatorname{Im} B_n - \gamma)}\right), \qquad \rho_n = |\omega_n|. \tag{1.8}$$

Let $\mathscr{A}$ denote the complex z-plane with the exception of the zeros of A_n $(n=0, 1,...)$. By (1.1) and the hypothesis $P_0 \neq 0$, we easily see that if z belongs to $\mathscr{A}$ and $P_{n+1}(z)=0$, then $P_n(z) \neq 0$. Therefore, according to (1.3), for any z belonging to $\mathscr{A}$, the polynomial $P_{n+1}(z) \neq 0$ if and only if:

$$t_n(z) \neq B_n(z). \tag{1.9}$$

Our method consists of exploiting this remark to determine zero-free regions for polynomials satisfying (1.1).

In order to study the distribution of zeros of a family produced by a recurrence relation with positive coefficients, we give our central result:

THEOREM 1.1. *Let $\{P_n\}_{n=0}^{N}$ be a sequence of polynomials of respective degrees n which satisfy the three-term recurrence relation with complex coefficients*:

$$P_{n+1}(z) = (b_n + b'_n z) P_n(z) - A_n(z) P_{n-1}(z) \qquad (n=0,..., N-1), \tag{1.10}$$

where $P_{-1} \equiv 0, P_0 \equiv p_0 \neq 0$; $b'_n > 0$ $(n=0, 1,..., N-1)$. Then this family of polynomials has no zeros in the region $\mathscr{P}_N$ defined by:

$$\mathscr{A}_N = \{z \in \mathbb{C} \mid A_n(z) \neq 0 \qquad (n=0, 1,..., N-1)\}, \tag{1.11}$$

$$\begin{aligned} 0 \leqslant n < N-1: f_n(z, d) &= \frac{|A_{n+1}(z)| + \operatorname{Re} A_{n+1}(z)}{2b'_n(\operatorname{Re} z - d)}, \\ \tilde{f}_n(z, d) &= \frac{|A_{n+1}(z)| - \operatorname{Re} A_{n+1}(z)}{2b'_n(\operatorname{Im} z - d)}, \end{aligned} \tag{1.12}$$

$$\begin{aligned} 0 \leqslant n < N: g_n(d) &= \operatorname{Re} b_n + b'_n d, \\ \tilde{g}_n(d) &= \operatorname{Im} b_n + b'_n d, \end{aligned} \tag{1.13}$$

$$\begin{aligned} I_{1,N} &= \left]\max_{0<n<N} -\frac{\operatorname{Re} b_n}{b'_n}, +\infty\right[\cap \left[-\frac{\operatorname{Re} b_0}{b'_0}, +\infty\right[, \\ I_{2,N} &= \left]-\infty, \min_{0<n<N} -\frac{\operatorname{Re} b_n}{b'_n}\right[\cap \left]-\infty, -\frac{\operatorname{Re} b_0}{b'_0}\right] \end{aligned} \tag{1.14}$$

$$\begin{aligned} I_{3,N} &= \left]\max_{0<n<N} -\frac{\operatorname{Im} b_n}{b'_n}, +\infty\right[\cap \left[-\frac{\operatorname{Im} b_0}{b'_0}, +\infty\right[, \\ I_{4,N} &= \left]-\infty, \min_{0<n<N} -\frac{\operatorname{Im} b_n}{b'_n}\right[\cap \left]-\infty, -\frac{\operatorname{Im} b_0}{b'_0}\right] \end{aligned} \tag{1.15}$$

$$\mathscr{P}_{1,N} = \bigcup_{d \in I_{1,N}} \left(\bigcap_{0 \leqslant n < N-1} \{z \in \mathscr{A}_N \,|\, f_n(z,d) \leqslant g_{n+1}(d), \operatorname{Re} z > d\} \right), \quad (1.16)$$

$$\mathscr{P}_{2,N} = \bigcup_{d \in I_{2,N}} \left(\bigcap_{0 \leqslant n < N-1} \{z \in \mathscr{A}_N \,|\, f_n(z,d) \geqslant g_{n+1}(d), \operatorname{Re} z < d\} \right), \quad (1.17)$$

$$\mathscr{P}_{3,N} = \bigcup_{d \in I_{3,N}} \left(\bigcap_{0 \leqslant n < N-1} \{z \in \mathscr{A}_N \,|\, \tilde{f}_n(z,d) \leqslant \tilde{g}_{n+1}(d), \operatorname{Im} z > d\} \right), \quad (1.18)$$

$$\mathscr{P}_{4,N} = \bigcup_{d \in I_{4,N}} \left(\bigcap_{0 \leqslant n < N-1} \{z \in \mathscr{A}_N \,|\, \tilde{f}_n(z,d) \geqslant \tilde{g}_{n+1}(d), \operatorname{Im} z < d\} \right), \quad (1.19)$$

$$\mathscr{P}_N = \bigcup_{1 \leqslant i \leqslant 4} \mathscr{P}_{i,N}. \quad (1.20)$$

Remark. According to the following proof, we can easily see that if:

$$b_n' < 0, \qquad n = 0, 1, ..., N-1,$$

then we have merely to change

(1.16) into:

$$\mathscr{P}_{1,N} = \bigcup_{d \in I_{2,N}} \left(\bigcap_{0 \leqslant n < N-1} \{z \in \mathscr{A}_N \,|\, f_n(z,d) \leqslant g_{n+1}(d), \operatorname{Re} z < d\} \right),$$

(1.17) into:

$$\mathscr{P}_{2,N} = \bigcup_{d \in I_{1,N}} \left(\bigcap_{0 \leqslant n < N-1} \{z \in \mathscr{A}_N \,|\, f_n(z,d) \geqslant g_{n+1}(d), \operatorname{Re} z > d\} \right),$$

(1.18) into:

$$\mathscr{P}_{3,N} = \bigcup_{d \in I_{4,N}} \left(\bigcap_{0 \leqslant n < N-1} \{z \in \mathscr{A}_N \,|\, \tilde{f}_n(z,d) \leqslant \tilde{g}_{n+1}(d), \operatorname{Im} z < d\} \right),$$

and

(1.19) into:

$$\mathscr{P}_{4,N} = \bigcup_{d \in I_{3,N}} \left(\bigcap_{0 \leqslant n < N-1} \{z \in \mathscr{A}_N \,|\, \tilde{f}_n(z,d) \geqslant \tilde{g}_{n+1}(d), \operatorname{Im} z > d\} \right).$$

Proof of Theorem 1.1. To find zero-free regions for polynomials satisfying (1.10) use, as mentioned, remark (1.9).

(1) *Case* $d \in I_{1,N}$. Let d be a parameter belonging to $I_{1,N}$:

$$d > \max_{0 < n < N} -\frac{\operatorname{Re} b_n}{b_n'}, d \geqslant -\frac{\operatorname{Re} b_0}{b_0'}. \quad (1.21)$$

Then, according to the definition of the function g_n (see (1.13)) we have:

$$0 < n < N\colon g_n(d) > 0;\ g_0(d) \geqslant 0. \tag{1.22}$$

Let z belong to the following set (cf. (1.16)):

$$\bigcap_{0 \leqslant n < N-1} \{z \in \mathscr{A}_N \mid f_n(z, d) \leqslant g_{n+1}(d), \operatorname{Re} z > d\} \tag{1.23}$$

and suppose, for the moment, that z is not a zero of any P_n. Then, since $\operatorname{Re} z > d$, the functions g_n satisfy:

$$0 \leqslant n < N, \qquad g_n(d) < \operatorname{Re} B_n(z), \tag{1.24}$$

where the polynomials B_n are defined by: $B_n(z) = b_n + b'_n z$.

We show by induction that

$$0 \leqslant n < N\colon \operatorname{Re} t_n(z) \leqslant g_n(d), \tag{1.25}$$

where the functions t_n are given by (1.4). If d belongs to $I_{1,N}$, then according to (1.22):

$$\operatorname{Re} t_0 = 0 \leqslant g_0(d).$$

Hence (1.25) obviously holds for $n = 0$. Now, let us suppose that for $u \neq 0$

$$\operatorname{Re} t_n(z) \leqslant g_n(d). \tag{1.26}$$

Since, according to (1.5), $t_{n+1}(z) = T_n(t_n(z))$, we have, putting in (1.8) $\gamma = g_n(d)$:

$$\operatorname{Re} t_{n+1}(z) \leqslant \operatorname{Re} \omega_n + \rho_n = f_n(z, d), \tag{1.27}$$

where the function f_n is given by (1.12). Hence, according to (1.23),

$$\operatorname{Re} t_{n+1}(z) \leqslant f_n(z, d) \leqslant g_{n+1}(d).$$

This completes the inductive establishing of (1.25). Therefore, according to (1.24), we can infer:

$$0 \leqslant n < N, \qquad \operatorname{Re} t_n(z) < \operatorname{Re} B_n(z). \tag{1.28}$$

We have seen, according to (1.2), that to define the quantity $t_n(z)$ we must suppose $P_n(z) \neq 0$. It is for this reason that in our proof we have had to suppose z is not a zero of any P_n ($n = 1, \dots, N-1$).

It is clear that the polynomial P_1 has no zero ($z_1 = -b_0/b'_0$) in the set defined by (1.23), since $d \geqslant -\operatorname{Re} b_0/b'_0$ and $\operatorname{Re} z > d$. Hence, we can say that the quantities t_0 and $t_1(z)$ exist for any z belonging to the set (1.23). Hence, if $0 \leqslant n < 2$, we have (1.28). Therefore, according to the remark (1.9), we have, for any z belonging to the set (1.23), $P_2(z) \neq 0$. Hence, for any z belonging to the set (1.23), the quantities $t_0, t_1(z)$ and $t_2(z)$ exist, and in the same way $P_3(z) \neq 0$ and so on.

So, we can conclude that for any d belonging to $I_{1,N}$, the set (1.23) contains no zero of any polynomial P_n ($n = 1,..., N$), which leads to the set (1.16).

(2) *Case* $d \in I_{2,N}$. The proof of this case is along the same lines as the proof in the case $d \in I_{1,N}$. Hence we merely sketch it.

We have just to change

$$(1.21) \text{ into } d < \min_{0<n<N} -\frac{\operatorname{Re} b_n}{b_n'}, d \leqslant -\frac{\operatorname{Re} b_0}{b_0'},$$

$$(1.22) \text{ into } 0 < n < N,\ g_n(d) < 0,\ g_0(d) \leqslant 0,$$

$$(1.23) \text{ into } \bigcap_{0 \leqslant n < N-1} \{z \in \mathscr{A}_N \mid f_n(z, d) \geqslant g_{n+1}(d), \operatorname{Re} z < d\},$$

$$(1.24) \text{ into } 0 \leqslant n < N,\ g_n(d) > \operatorname{Re} B_n(z),$$

$$(1.25) \text{ into } 0 \leqslant n < N, \operatorname{Re} t_n(z) \geqslant g_n(d),$$

$$(1.26) \text{ into } n \neq 0, \operatorname{Re} t_n(z) \geqslant g_n(d),$$

$$(1.27) \text{ into } \operatorname{Re} t_{n+1}(z) \geqslant \operatorname{Re} \omega_n - \rho_n = f_n(z, d)$$

and

$$(1.28) \text{ into } 0 \leqslant n < N, \operatorname{Re} t_n(z) > \operatorname{Re} B_n(z).$$

So, in this case, we can also conclude that $\mathscr{P}_{2,N}$ given by (1.17) contains no zero of any P_n ($n = 1,..., N$).

(3) *Case* $d \in I_{3,N}$. The proof of this case is the same as the proof of the case $d \in I_{1,N}$. Here, we have just to change, in (1.8), $\omega_n = A_{n+1}/2(\operatorname{Re} B_n - \gamma)$ into $\omega_n = -iA_{n+1}/2(\operatorname{Im} B_n - \gamma)$, the functions f_n and g_n into $\tilde{f}_n$ and $\tilde{g}_n$ given, respectively, by (1.12) and (1.13), and in step (1) to change Re into Im. Hence, $\mathscr{P}_{3,N}$ given by (1.18) contains no zero of any polynomial P_n ($n = 1,..., N$).

(4) *Case* $d \in I_{4,N}$. This case corresponds to case (3) as case (2) corresponds to $d \in I_{1,N}$. Hence, we again conclude that the set $\mathscr{P}_{4,N}$ defined by (1.19) contains no zero of any polynomial P_n ($n = 1,..., N$).

So, we can infer that the region $\mathscr{P}_N$ given by (1.20) does not contain any zero of the family $\{P_n\}_{n=0}^N$. ∎

2. Special Cases

In this section, we give two new results on the location of zeros of some special polynomials.

We locate the zeros of families of polynomials which satisfy a three-term recurrence relation with positive coefficients. As we said before, our principal tool will be Theorem 1.1.

2.1. *Conic Theorem*

The following result complements Saff and Varga "parabola theorem" [5]. We shall see that the following regions are bounded by conic curves; it is for this reason that we name our result "conic theorem."

Theorem 2.1 (Conic Theorem). *Let* $\{P_n\}_{n=0}^N$ *be a sequence of polynomials of respective degrees n which satisfy the three-term recurrence relation*:

$$P_{n+1}(z) = (b_n + b'_n z) P_n(z) - a_n z P_{n-1}(z) \qquad (n = 0, \dots, N-1), \quad (2.1)$$

where: $P_{-1} \equiv 0, P_0 \equiv p_0 \neq 0$, *and for all integers n* $(n = 0, \dots, N-1)$ *the* a_n *and* b'_n *are positive numbers, while the* b_n *are real numbers. Set*:

$$\varphi(z, d) = \frac{|z| + \operatorname{Re} z}{2(\operatorname{Re} z - d)}, \qquad \tilde{\varphi}(z, d) = \frac{|z| - \operatorname{Re} z}{2(\operatorname{Im} z - d)}, \quad (2.2)$$

$$0 < n < N\colon \psi_n(d) = \frac{b'_{n-1}}{a_n} (b_n + b'_n d), \qquad \tilde{\psi}_n(d) = \frac{b'_{n-1} b'_n}{a_n} d, \quad (2.3)$$

and

$$\Gamma_{1,N} = \bigcup_{d \geqslant -b_0/b'_0, d > \max_{0<n<N}(-b_n/b'_n)} \{z \in \mathbb{C} \mid \varphi(z, d) \leqslant \min_{0<n<N} \psi_n(d), \operatorname{Re} z > d\}, \quad (2.4)$$

$$\Gamma_{2,N} = \bigcup_{d \leqslant -b_0/b'_0, d < \min_{0<n<N}(-b_n/b'_n)} \{z \in \mathbb{C} \mid \varphi(z, d) \geqslant \max_{0<n<N} \psi_n(d), \operatorname{Re} z < d\}, \quad (2.5)$$

$$\Gamma_{3,N} = \bigcup_{d > 0} \{z \in \mathbb{C} \mid \tilde{\varphi}(z, d) \leqslant \min_{0<n<N} \tilde{\psi}_n(d), \operatorname{Im} z > d\}, \quad (2.6)$$

$$\Gamma_{4,N} = \bigcup_{d<0} \{z \in \mathbb{C} \mid \tilde{\varphi}(z, d) \geqslant \max_{0<n<N} \tilde{\psi}_n(d), \operatorname{Im} z < d\}. \tag{2.7}$$

Then the region Γ_N *defined by*:

$$\Gamma_N = \bigcup_{1 \leqslant i \leqslant 4} \Gamma_{i,N} \tag{2.8}$$

contains no zero of the family $\{P_n\}_{n=0}^N$.

Remark. In (2.4), for a fixed parameter: $d \geqslant -b_0/b_0', d > \max_{0<n<N} -b_n/b_n'$, the set $\{z \in \mathbb{C} \mid \varphi(z, d) \leqslant \min_{0<n<N} \psi_n(d), \operatorname{Re} z > d\}$ is bounded by an elliptic curve when $\min_{0<n<N} \psi_n(d) < 1$ and by a parabolic curve when $\min_{0<n<N} \psi_n(d) \geqslant 1$. In (2.5), for a fixed d: $d \leqslant -b_0/b_0', d < \min_{0<n<N} -b_n/b_n'$, the set $\{z \in \mathbb{C} \mid \varphi(z, d) \geqslant \max_{0<n<N} \psi_n(d), \operatorname{Re} z < d\}$ is bounded by a branch of an hyperbola, and in (2.6), for a fixed $d > 0$, the set $\{z \in \mathbb{C} \mid \tilde{\varphi}(z, d) \leqslant \min_{0<n<N} \tilde{\psi}_n(d), \operatorname{Im} z > d\}$ is also bounded by a branch of an hyperbola. The set $\Gamma_{4,N}$ defined by (2.7) is, in fact, symmetric to the set $\Gamma_{3,N}$ given by (2.6) with respect to the real axis.

Proof of Theorem 2.1. Theorem 2.1 is an immediate consequence of Theorem 1.1. Indeed in this case, according to (1.14) and (1.15):

$$I_{1,N} = \left]\max_{0<n<N} -\frac{b_n}{b_n'}, +\infty\right[\cap \left[-\frac{b_0}{b_0'}, +\infty\right[,$$

$$I_{2,N} = \left]-\infty, \min_{0<n<N} -\frac{b_n}{b_n'}\right[\cap \left]-\infty, -\frac{b_0}{b_0'}\right]$$

$$I_{3,N} =]0, +\infty[, I_{4,N} =]-\infty, 0[.$$

Putting in (1.10): $A_n(z) = a_n z$ $(n = 0, 1, ..., N-1)$, we have, according to (1.11): $\mathscr{A}_N = \mathbb{C} - \{0\}$.

Let us suppose for a moment that for all n: $b_n \neq 0$. Then (2.1) and the hypothesis $P_0 \equiv p_0 \neq 0$ show that $z = 0$ is not a zero of any P_n. Hence, in (2.4)–(2.7), $\mathscr{A}_N = \mathbb{C}$. Now, let us suppose that one or more $b_n = 0$. According to the condition on the parameter d, the sets (2.4)–(2.7) do not contain the point $z = 0$. So we can take, also in this case, $\mathscr{A}_N$ to be $\mathbb{C}$.

In (1.16)–(1.19), according to the hypothesis on the numbers a_n, b_n and b_n', the inequalities $f_n(z, d) \leqslant g_{n+1}(d), f_n(z, d) \geqslant g_{n+1}(d)$, $\tilde{f}_n(z, d) \leqslant \tilde{g}_{n+1}(d)$ and $\tilde{f}_n(z, d) \geqslant \tilde{g}_{n+1}(d)$ are, respectively, equivalent to $\varphi(z, d) \leqslant \psi_{n+1}(d)$, $\varphi(z, d) \geqslant \psi_{n+1}(d)$, $\tilde{\varphi}(z, d) \leqslant \tilde{\psi}_{n+1}(d)$ and $\tilde{\varphi}(z, d) \geqslant \tilde{\psi}_{n+1}(d)$, where the functions $\varphi, \tilde{\varphi}, \psi_{n+1}$ and $\tilde{\psi}_{n+1}$ are given by (2.2) and (2.3). Hence the sets $\mathscr{S}_{i,N}$ $(1 \leqslant i \leqslant 4)$, given by (1.16)–(1.19), obviously become in the present case $\Gamma_{i,N}$ $(1 \leqslant i \leqslant 4)$, respectively, which implies $\mathscr{P}_N = \Gamma_N$ (see (1.20) and (2.8)). ∎

This theorem allows one to study (when we put $b_n = 1$, $0 \leqslant n < N$) the distribution of the zeros and poles of Padé approximants to some functions [4], and to locate the zeros of some special polynomials.

2.2. *Location of the Zeros of Orthogonal Polynomials*

Here we give a new method which allows us to locate the zeros of any finite family of classical orthogonal polynomials.

THEOREM 2.2. *Let $\{P_n\}_{n=0}^{N}$ be a sequence of polynomials of respective degrees n which satisfy the three-term recurrence relation*:

$$P_{n+1}(z) = (b_n + b'_n z)\, P_n(z) - a_n P_{n-1}(z) \qquad (n = 0, 1, \ldots, N-1), \quad (2.9)$$

where: $P_{-1} \equiv 0, P_0 \equiv p_0 \neq 0$, *and for all integer n* $(0 \leqslant n < N)$ *the* a_n *and* b'_n *are positive numbers, while the* b_n *are real numbers. Set*:

$$\forall x \leqslant x_1 = \min_{0 \leqslant n < N} -\frac{b_n}{b'_n} : m(x) = \min_{0 < n < N} \frac{a_n}{b'_{n-1}} (b_n + b'_n x)^{-1}, \quad (2.10)$$

$$\forall x \geqslant x_2 = \max_{0 \leqslant n < N} -\frac{b_n}{b'_n} : M(x) = \max_{0 < n < N} \frac{a_n}{b'_{n-1}} (b_n + b'_n x)^{-1}. \quad (2.11)$$

Then the polynomial P_n $(n = 1, \ldots, N)$ *has all its zeros in the open interval* I_N *defined by*:

$$I_N = \left]\max_{x \leqslant x_1} (x + m(x)), \min_{x \geqslant x_2} (x + M(x))\right[. \quad (2.12)$$

Remark. This theorem also allows one to study the evolution with N of the support of a positive measure defining the orthogonal polynomials.

Proof of Theorem 2.2. This theorem is also an immediate consequence of Theorem 1.1.

First let us observe, by the definition (1.11) of $\mathscr{A}_N$, that

$$\mathscr{A}_N = \mathbb{C}, \quad (2.13)$$

and that the function m given by (2.10) has its maximum at a point belonging to $I_{2,N}$ of (1.14), because $\lim_{I_{2,N} \ni x \to \min_{0<n<N}(-b_n/b'_n)} m(x) = -\infty$. Similarly, the function M given by (2.11) has its minimum at a point

belonging to $I_{1,N}$ of (1.14) as, in this case, $\lim_{I_{1,N}\ni x\to\max_{0<n<N}(-b_n/b'_n)} M(x) = +\infty$. Hence we can write:

$$\operatorname*{Inf}_{x\in I_{1,N}} (x+M(x)) = \min_{x\geqslant x_2} (x+M(x)),$$

$$\operatorname*{Sup}_{x\in I_{2,N}} (x+m(x)) = \max_{x\leqslant x_1} (x+m(x)), \tag{2.14}$$

where the quantities x_1 and x_2 are given, respectively, by (2.10) and (2.11).

According to the hypothesis on the numbers a_n, b_n and b'_n, and (2.13) and (2.14), the sets $\mathscr{P}_{1,N}$, $\mathscr{P}_{2,N}$, $\mathscr{P}_{3,N}$ and $\mathscr{P}_{4,N}$ given by (1.16)–(1.19) become obviously:

$$\mathscr{P}_{1,N} = \{z\in\mathbb{C} \mid \operatorname{Re} z \geqslant \operatorname*{Inf}_{d\in I_{1,N}} (d+M(d))\}$$
$$= \{z\in\mathbb{C} \mid \operatorname{Re} z \geqslant \min_{d\geqslant x_2} (d+M(d))\},$$
$$\mathscr{P}_{2,N} = \{z\in\mathbb{C} \mid \operatorname{Re} z \leqslant \operatorname*{Sup}_{d\in I_{2,N}} (d+m(d))\}$$
$$= \{z\in\mathbb{C} \mid \operatorname{Re} z \leqslant \max_{d\leqslant x_1} (d+m(d))\},$$
$$\mathscr{P}_{3,N} = \{z\in\mathbb{C} \mid \operatorname{Im} z > 0\}, \mathscr{P}_{4,N} = \{z\in\mathbb{C} \mid \operatorname{Im} z < 0\}.$$

So, we can infer that the family $\{P_n\}_{n=0}^{N}$ has all its zeros in the set: $\mathbb{C} - \mathscr{P}_N$, where $\mathscr{P}_N$ is given by (1.20), which implies the conclusion of the theorem. ∎

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